Effective algebraicity for solutions of systems of functional equations with one catalytic variable

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Abstract

We study systems of $n \geq 1$ discrete differential equations of order $k \geq 1$ in one catalytic variable and provide a constructive and elementary proof of algebraicity of their solutions. This yields effective bounds and a systematic method for computing the minimal polynomials. Our approach is a generalization of the pioneering work by Bousquet-Mélou and Jehanne (2006).

1 Introduction

Numerous combinatorial enumeration problems reduce to the study of functional equations which can be solved by a uniform method introduced by Bousquet-Mélou and Jehanne in the seminal work [4]. These functional equations, usually called discrete differential equations (DDEs) with one catalytic variable, involve a bivariate generating function $F \in \mathbb{Q}[u][[t]]$ associated to the enumeration problem, and are of the form

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u), \tag{1}$$

where $k \in \mathbb{N}$ (called order of the DDE), f and Q are polynomials, and (for some $a \in \mathbb{Q}$, usually 0 or 1) Δ_a^ℓ is the ℓth iteration of the operator $\Delta_a : \mathbb{Q}[u][[t]] \to \mathbb{Q}[u][[t]]$ defined by

$$\Delta_a F(t,u) := \frac{F(t,u) - F(t,a)}{u-a}.$$

In their paper, Bousquet-Mélou and Jehanne designed a "non-linear kernel method" which allows to prove that the unique solution of (1) is always an algebraic function over $\mathbb{Q}(t,u)$. Significantly in practice, this approach yields a strategy/algorithm for finding the minimal polynomial of the specialization $F(t,a)$ and of the bivariate series $F(t,u)$.

The main contribution of the present paper is the generalization of this method to the case of *systems of discrete differential equations*. More precisely, we shall prove the following theorem (for any field $\mathbb{K}$ of characteristic 0):

Theorem 1. *Let $n, k \geq 1$ be integers and let $f_1, \dots, f_n \in \mathbb{K}[u]$ and $Q_1, \dots, Q_n \in \mathbb{K}[y_1, \dots, y_{n(k+1)}, t, u]$ be polynomials. Set $\nabla^k F := F, \Delta_a F, \dots, \Delta_a^k F$. Then the system of equations*

$$\begin{cases} (\mathbf{E_{F_1}})\colon\ F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ (\mathbf{E_{F_n}})\colon\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) \end{cases} \tag{2}$$

admits a unique vector of solutions $(F_1, \dots, F_n) \in \mathbb{K}[u][[t]]^n$, and all its components are algebraic functions over $\mathbb{K}(t, u)$.

The key idea, analogous to the one in [4], for proving this theorem is to define the correct deformation of (2) that ensures the applicability of a multi-dimensional analog of the "non-linear kernel method". Stated explicitly, we show in Lemma 1 that after deforming the equations as in (6), the polynomial in u defined by the determinant of the Jacobian matrix associated to the numerator equations in (2) (considered with respect to the F_i) has exactly nk solutions in an extension of the ring $\cup_{d \geq 1} \overline{\mathbb{K}}[[t^{1/d}]]$. Then, after a process of "duplication of variables", we construct a zero-dimensional and radical polynomial ideal, a non-trivial element of which must be the seeked annihilating polynomial. The most difficult step consists in proving the invertibility of a certain Jacobian matrix (Lemma 3 and Lemma 4) in order to justify the zero-dimensionality. We remark that an alternative, and possibly more practical, strategy is to reduce the initial system to a single functional equation. Our Proposition 1 ensures that such a reduction preserves the roots guaranteed in the deformation step, however, as we will show in Section 4, this method is not guaranteed to produce a zero-dimensional polynomial ideal in the end.

Similarly to the work by Bousquet-Mélou and Jehanne, our proof is effective, in the sense that it produces an algorithm for finding the minimal polynomials of the power series of interest. Moreover, we can deduce a bound on the algebraicity degree of any F_i. When $\mathbb{K} = \mathbb{Q}$, we can also bound the arithmetic complexity of our algorithm, that is the number of operations $(+, -, \times, \div)$ performed in $\mathbb{Q}$. Denoting by $\mathsf{totdeg}(P)$ the total degree of a multivariate polynomial P, we obtain the following:

Theorem 2. *In the setting of Theorem 1, let $(F_1, \dots, F_n) \in \mathbb{K}[u][[t]]^n$ be the vector of solutions. Let $\delta := \max(\deg(f_1), \dots, \deg(f_n), \mathsf{totdeg}(Q_1), \dots, \mathsf{totdeg}(Q_n))$. Then the algebraicity degree of each $F_i(t, u)$ over $\mathbb{K}(t, u)$ is bounded by $(nk\delta)^{4n^2k^2}$. Moreover, if $\mathbb{K} = \mathbb{Q}$, there exists an algorithm computing the minimal polynomial of any $F_i(t, a)$ in $(2nk\delta)^{O(nk)}$ arithmetic operations.*

Discrete differential equations are ubiquitous in enumerative combinatorics [5, 6, 11]. Systems of DDEs also appear in a variety of different contexts throughout combinatorics, for instance for hard particles on planar maps [4, §5.4], inhomogeneous lattice paths [7], or certain orientations with n edges [2, §5]. The usual strategy for solving these systems of equations is to try to reduce a given system to a scalar equation and then apply the method of Bousquet-Mélou and Jehanne. This approach is usually ad-hoc and needs to exploit additional structure of the system. Moreover, since the reduced equation is in general not of the form (1) anymore, the theory of [4] is not guaranteed to work.

In the literature there exist two methods to overcome these theoretical issues. First, a deep theorem in commutative algebra by Popescu [9], so-called "nested Artin approximation", guarantees that equations of the form (2) always admit an algebraic solution. Note that the nested condition is automatically satisfied in this case and that the uniqueness of the solution is obvious. The drawback of using Popescu's theorem, however, is that its proof is highly non-constructive and can only be applied as a "black box", whereas in practice one is often interested in the explicit minimal polynomials annihilating the solutions. Secondly, the frequent case when all polynomials $Q_1, \dots, Q_n$ in (2) are linear functions was effectively solved in the recent FPSAC article [7] by Buchacher and Kauers who designed a "multi-dimensional kernel method". From this viewpoint, it is safe to say that our contribution is a common generalization of central results by Bousquet-Mélou and Jehanne [4] and Buchacher and Kauers [7], and at the same time an effective and elementary proof of a special case of Popescu's theorem [9].

From the examples of systems of discrete differential equations we already mentioned, we shall highlight the following two in more detail.

Example 1. The following system of DDEs for the generating function of certain planar orientations was considered in [2, Eq.(27)] and solved in the same work:

$$\begin{cases} (\mathbf{E}_{\mathbf{F}_1})\textbf{:}\ F_1(t,u) = 1 + t\cdot\left(u + 2uF_1(t,u)^2 + 2uF_2(t,1) + u\frac{F_1(t,u)-uF_1(t,1)}{u-1}\right), \\ (\mathbf{E}_{\mathbf{F}_2})\textbf{:}\ F_2(t,u) = t\cdot\left(2uF_1(t,u)F_2(t,u) + uF_1(t,u) + uF_2(t,1) + u\frac{F_2(t,u)-uF_2(t,1)}{u-1}\right). \end{cases} \tag{3}$$

From our perspective, (3) has the advantage that it does not require any deformation and, as we will show in Section 2, it can be solved fast by a direct application of our method. It is thus a good illustration of the simplest non-trivial case of our approach.

Example 2. This example of a system of DDEs modelling a particular case of hard particles on planar maps was introduced and solved in [4, Section 11]:

$$\begin{cases} (\mathbf{E}_{\mathbf{F}_1})\textbf{:}\ F_1(t,u) = F_2(t,u) + tu^2F_1(t,u)^2 + tu\frac{uF_1(t,u)-F_1(t,1)}{u-1}, \\ (\mathbf{E}_{\mathbf{F}_2})\textbf{:}\ F_2(t,u) = 1 + tsuF_1(t,u)F_2(t,u) + tsu\frac{F_2(t,u)-F_2(t,1)}{u-1}. \end{cases} \tag{4}$$

As we will explain in Section 4, in order to apply our method directly, the deformation step (6) is necessary.

The structure of the paper is as follows: In Section 2 we explain our method in the case of two equations of order one under the genericity assumption that no deformation is necessary. We summarize the method in an algorithm and showcase it explicitly on Example 1. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In the last Section 4 we briefly explore an improvement to our approach which in theory has a better algorithmic complexity but which requires a new genericity assumption. We also discuss possible future works.

2 The case of two generic equations of first order

Before proving our main theorem in Section 3, we introduce our method in the situation of two equations of order 1 and under a genericity assumption on the input system.

Starting with (2), we first multiply $(\mathbf{E}_{\mathbf{F}_1})$ and $(\mathbf{E}_{\mathbf{F}_2})$ by $(u-a)^{m_1}$ and $(u-a)^{m_2}$ respectively (for $m_1, m_2 \in \mathbb{N}$) in order to obtain a system with polynomial coefficients in u. By a slight abuse of notation, we shall still write $(\mathbf{E}_{\mathbf{F}_1})$ and $(\mathbf{E}_{\mathbf{F}_2})$ for those equations. Note that this system induces polynomials E_1, E_2 in $\mathbb{K}[x_1, x_2, z_0, z_1, t, u]$ whose specializations to $x_1 = F_1(t,u), x_2 = F_2(t,u), z_0 = F_1(t,a), z_1 = F_2(t,a)$ are zero.

Example 1 (cont.). Multiplying $(\mathbf{E}_{\mathbf{F}_1})$ and $(\mathbf{E}_{\mathbf{F}_2})$ in Example 1 by $u-1$ gives

$$\begin{cases} E_1 = (1-x_1)\cdot(u-1) + t\cdot(2u^2x_1^2 - u^2z_0 + 2u^2z_1 - 2ux_1^2 + u^2 + ux_1 - 2uz_1 - u), \\ E_2 = x_2\cdot(1-u) + t\cdot(2u^2x_1x_2 + u^2x_1 - 2ux_1x_2 - ux_1 + ux_2 - uz_1). \end{cases}$$

In the spirit of [4], we now take the derivative of both equations with respect to u:

$$\begin{pmatrix} \partial_{x_1}E_1 & \partial_{x_2}E_1 \\ \partial_{x_1}E_2 & \partial_{x_2}E_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_u F_1 \\ \partial_u F_2 \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \partial_u E_2 \end{pmatrix} = 0. \tag{5}$$

Define $\mathsf{Det} := \partial_{x_1}E_1 \cdot \partial_{x_2}E_2 - \partial_{x_1}E_2 \cdot \partial_{x_2}E_1 \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]$ to be the determinant of the square matrix above. One can show that

$$\mathsf{Det}(F_1(t,u), F_2(t,u), F_1(t,a), F_2(t,a), t, u) \in \mathbb{K}[[t]][[u]]$$

admits either $0, 1$ or 2 distinct non-zero solutions $u = U(t) \in \cup_{d\geq 1}\overline{\mathbb{K}}[[t^{\frac{1}{d}}]] =: \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$. We assume now that there exist 2 such solutions $U_1, U_2 \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$; we prove in Section 3 that it is always the case up to the deformation (6).

Exploiting the idea of [7], we now define $v := (\partial_{x_1}E_2 \quad -\partial_{x_1}E_1)$ and plug $u = U_1$ into v and into (5). Note that v is an element of the left-kernel of the square matrix in (5) mod $\mathsf{Det}(F_1(t,u), F_2(t,u), F_1(t,a), F_2(t,a), t, u)$. After multiplication of (5) by v on the left, we find a new polynomial relation between $F_1(t,U_i), F_2(t,U_i), F_1(t,a), F_2(t,a), t$ and U_i, namely $\partial_{x_1}E_1 \cdot \partial_u E_2 - \partial_{x_1}E_2 \cdot \partial_u E_1 = 0$ when evaluated at $u = U_i$. We denote this polynomial by $P \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]$.

Define the polynomial system $\mathcal{S} := (E_1, E_2, \mathsf{Det}, P) \in \mathbb{K}[t][x_1, x_2, z_0, z_1, u]^4$. It admits the non-trivial solutions $(F_1(t,U_i), F_2(t,U_i), F_1(t,a), F_2(t,a), U_i) \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]^5$, for $i \in \{1,2\}$.

Example 1 (cont.). Continuing Example 1, we find

$$\begin{cases} \mathsf{Det} = (4tu^2x_1 - 4tux_1 + tu - u + 1)(2tu^2x_1 - 2tux_1 + tu - u + 1), \\ P = -2tx_1x_2 - tx_1 + tx_2 - tz_1 - x_2 + P_1\cdot u + P_2\cdot u^2 + P_3\cdot u^3, \end{cases}$$

where P_1, P_2, P_3 are explicit (but relatively big) polynomials in $\mathbb{Q}[x_1, x_2, z_0, z_1, t]$.

Now, generalizing naturally the steps of [4], we define for $i \in \{0,1\}$ the polynomial systems $\mathcal{S}_i := \mathcal{S}(x_{2i+1},\ x_{2i+2},\ z_0,\ z_1,\ t,\ u_i)$ by "duplicating" variables. If $(\mathbf{E_{F_1}})$ and $(\mathbf{E_{F_2}})$ are "generic", the ideal $\langle \mathcal{S}_0, \mathcal{S}_1, m\cdot(u_1-u_2)-1\rangle$ has dimension 0 over $\mathbb{K}(t)$ and hence it is enough to compute a non-zero element of $\langle \mathcal{S}_0, \mathcal{S}_1, m\cdot(u_1-u_2)-1\rangle \cap \mathbb{K}[z_0,t]$ to find an annihilating polynomial of $z_0 = F_1(t,a)$.

Example 1 (cont.). Continuing Example 1, we compute[1] a generator of the polynomial ideal $\langle \mathcal{S}_0, \mathcal{S}_1, m\cdot(u_1-u_2)-1\rangle \cap \mathbb{Q}[z_0,t]$. It has degree 13 in z_0 and 14 in t. In particular, it contains in its factors the minimal polynomial of $F_1(t,a)$ given by $(64z_0^3+48z_0^2-15z_0+1)t^3+(-72z_0^2+9z_0+27)t^2+(2z_0^2+19z_0-19)t-z_0+1$.

We summarize the presented algorithm in the above more compact form.

Algorithm 1: Solving generic systems of two fixed point equations of first order.

Input: A "generic" system of two DDEs $(\mathbf{E_{F_1}})$, $(\mathbf{E_{F_2}})$ of order 1.

Output: A non-zero $R \in \mathbb{K}[z_0,t]$ annihilating $F_1(t,a)$.

Replace $(\mathbf{E_{F_1}})$ and $(\mathbf{E_{F_2}})$ by their respective numerators and denote by E_1 and E_2 the associated polynomials in $\mathbb{K}[x_1,x_2,z_0,z_1,t,u]$.

Compute $\mathsf{Det} := \partial_{x_1}E_1 \cdot \partial_{x_2}E_2 - \partial_{x_1}E_2 \cdot \partial_{x_2}E_1$ and $P := \partial_{x_1}E_1 \cdot \partial_u E_2 - \partial_{x_1}E_2 \cdot \partial_u E_1$.

Set $\mathcal{S} := (E_1, E_2, \mathsf{Det}, P) \subset \mathbb{K}[x_1,x_2,z_0,z_1,t,u]$.

For $0 \le i \le 1$, define $\mathcal{S}_i := \mathcal{S}(x_{2i+1},\ x_{2i+2},\ z_0,\ z_1,\ t,\ u_i)$.

Return a non-zero element of $\langle \mathcal{S}_0, \mathcal{S}_1, m\cdot(u_1-u_2)-1\rangle \cap \mathbb{K}[z_0,t]$.

As already stated, for degenerate inputs, any number of equations of any order, Section 3 ensures that we can always use Algorithm 1 up to a deformation.

We remark that if the strategy above is applied in the case of a single equation of first order $F_1 = f(u) + t\cdot Q_1(F_1, \Delta_a F_1, t, u)$, the presented method simplifies to the classical algorithm in [4] relying on studying the ideal $\langle E_1, \partial_{x_1}E_1, \partial_u E_1\rangle$. Stated explicitly, $\partial_{x_1}E_1$ plays the role of Det and $\partial_u E_1$ plays the role of P (as we can take here $v = 1$).

3 Proofs of Theorem 1 and Theorem 2

We start by proving Theorem 1. As explained before, the statement and proof can be seen as a generalization of [4, Theorem 3] and [7, Theorem 2], so several steps are done analogously. Without loss of generality we assume that $a = 0$ and set $\Delta := \Delta_0$.

Denote by $m_1, \ldots, m_n$ the least positive integers greater or equal than k such that multiplying $(\mathbf{E_{F_i}})$ in (2) by u^{m_i} gives a polynomial equation in u. Set $\beta := \lfloor 2M/k \rfloor$ and $\alpha := n^2k\cdot(\beta+1)+nM$, where $M := m_1 + \cdots + m_n$.

We will now define a deformation of the system of equations (2). For this purpose, let ϵ be a new variable, set $\mathbb{L} := \mathbb{K}(\epsilon)$, and let $(\gamma_{i,j})_{1\le i,j\le n}$ be defined by $\gamma_{i,i} = i^k$ and

[1] All computations in this paper have been performed using msolve [1]

$\gamma_{i,j} = t^{\beta}$ for $i \neq j$. Then consider the following system:

$$\begin{cases} \mathbf{(E_{G_1})}: & G_1 = f_1(u) + t^{\alpha} \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot \Delta^k G_i, \\ \quad\vdots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \mathbf{(E_{G_n})}: & G_n = f_n(u) + t^{\alpha} \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot \Delta^k G_i. \end{cases} \tag{6}$$

The fixed point nature of these equations still implies that there exists a unique solution $(G_1, \ldots, G_n) \in \mathbb{L}[u][[t]]^n$. Remark that the equalities $F_i(t^{\alpha}, u) = G_i(t, u, 0)$ relate the formal power series solutions of (2) and of (6). Hence, showing that each G_i is algebraic over $\mathbb{L}(t, u)$ is enough for proving Theorem 1. Moreover, as we will see later, the algebraicity of each G_i follows from the algebraicity of $G_1(0), \ldots, \partial_u^{k-1} G_1(0)$, $\ldots, G_n(0), \ldots, \partial_u^{k-1} G_n(0)$. Here, and in what follows, we shall use the short notation $G_i(u) \equiv G_i(t, u, \epsilon_1, \ldots, \epsilon_n)$, $\partial_0 G_i(u) \equiv G_i(u), G_i(0), \partial_u G_i(0), \ldots, \partial_u^{k-1} G_i(0)$ and $A(u) \equiv A(\partial_0 G_1, \ldots, \partial_0 G_n, t, u)$ for any polynomial $A \in \mathbb{L}[X_1, \ldots, X_n, t, u]$ with $X_j := x_j, z_{k(j-1)}, \ldots, z_{kj-1}$.

Let us define $Y_{i,0} := x_i$ and $Y_{i,j} := (x_i - z_{k(i-1)} - \cdots - \frac{u^{j-1}}{(j-1)!} z_{k(i-1)+j-1})/u^j$ for $j \geq 1$. With these definitions, (6) is equivalent to the following system of polynomial equations

$$\begin{cases} E_1 := u^{m_1} \cdot (f_1(u) - x_1 + t^{\alpha} \cdot Q_1(Y_{1,0}, \ldots, Y_{n,k}, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot Y_{i,k} = 0, \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ E_n := u^{m_n} \cdot (f_n(u) - x_n + t^{\alpha} \cdot Q_n(Y_{1,0}, \ldots, Y_{n,k}, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot Y_{i,k} = 0. \end{cases} \tag{7}$$

Like in (5), we take the derivative with respect to u of these equations and find

$$\begin{pmatrix} \partial_{x_1} E_1 & \ldots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \ldots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u G_1 \\ \vdots \\ \partial_u G_n \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \tag{8}$$

Let $\mathsf{Det} \in \mathbb{L}[X_1, \ldots, X_n, t][u]$ be the determinant of the square matrix $(\partial_{x_i} E_j)_{1 \leq i,j \leq n}$ above. The following lemma on the number of distinct solutions to $\mathsf{Det}(u) = 0$ is the first main step in our proof.

Lemma 1. $\mathsf{Det}(u) = 0$ *admits exactly* nk *distinct non-zero solutions* $U_1, \ldots, U_{nk} \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]$.

Proof. Note that we have

$$\mathsf{Det}(u) = \det \begin{pmatrix} -u^{m_1} + t\epsilon^k \gamma_{1,1} u^{m_1 - k} & \cdots & t\epsilon^k \gamma_{1,n} u^{m_1 - k} \\ \vdots & \ddots & \vdots \\ t\epsilon^k \gamma_{n,1} u^{m_n - k} & \cdots & -u^{m_n} + t\epsilon^k \gamma_{n,n} u^{m_n - k} \end{pmatrix} + O(t^{\alpha} u^{M - nk}).$$

For every i we first divide the i^{th} row by $u^{m_i - k}$. Then, using the definition of $\gamma_{i,j}$ and $\alpha, \beta \geq n$, we see that the matrix above becomes diagonal mod t^{n+1} and its determinant mod t^{n+1} simplifies to $\prod_{j=1}^n (-u^k + t\epsilon^k j^k)$ mod t^{n+1}. So, by the Newton-Raphsen algorithm, we find nk distinct solutions $u = U_1(t), \ldots, U_{nk}(t)$ whose first terms are given by $\zeta^\ell \cdot t^{\frac{1}{k}} \cdot \epsilon + O(t^{\frac{2}{k}}), \dots, \zeta^\ell \cdot n \cdot t^{\frac{1}{k}} \cdot \epsilon + O(t^{\frac{2}{k}}) \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]$, for ζ a k-primitive root of unity and for all $1 \le \ell \le k$. Finally, note that the constant coefficient in t of $\prod_{j=1}^n (-u^k + t\epsilon^k j^k)$ has degree nk so by [4, Theorem 2] there cannot be more than nk solutions to $\mathsf{Det}(u) = 0$ in $\overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]$. $\square$

Now, let P be the determinant of the square matrix $(\partial_{x_i} E_j)_{1 \le i,j \le n}$ where the last row $(\partial_{x_n} E_1, \dots, \partial_{x_n} E_n)$ is replaced by $(\partial_u E_1, \dots, \partial_u E_n)$. It is easy to see with standard linear algebra arguments that if $\mathsf{Det} = 0$ then (8) implies that $P = 0$. We thus define the polynomial system $(\mathcal{S}) \coloneqq (E_1, \dots, E_n, \mathsf{Det}, P)$ in $\mathbb{L}[t][X_1, \dots, X_n, u]$. We see that $(\mathcal{S})$ is a system with exactly $n+2$ equations and $nk + n + 1$ variables (t and ϵ are parameters). We wish to construct a zero-dimensional ideal, so we introduce the *duplicated system* $(\mathcal{S}_{\mathsf{dup}}) \coloneqq (\mathcal{S}_1, \dots, \mathcal{S}_{nk})$, defined in the duplicated ring $\mathbb{K}[x_1, \dots, x_{n^2k}, z_0, \dots, z_{nk-1}, u_1, \dots, u_{nk}, t, \epsilon]$. This system is built from $nk(n+2)$ equations and $nk(n+2)$ variables.

The following lemma is proven in [3, Lemma 2.10] as a consequence of Hilbert's Nullstellensatz and [8, Theorem 16.19]:

Lemma 2. *Assume that the Jacobian matrix $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ of $(\mathcal{S}_{\mathsf{dup}})$, considered with respect to the variables $x_1, \dots, x_n, u_1, \dots, x_{n^2k-n}, \dots, x_{n^2k}, u_{nk}, z_0, \dots, z_{nk-1}$ is invertible at the point*

$$\mathcal{P} = (G_1(U_1), \dots, G_n(U_1), U_1, \dots, G_1(U_{nk}), \dots, G_n(U_{nk}), U_{nk}, G_1(0), \dots, \partial_u^{k-1} G_1(0), \dots, G_n(0), \dots, \partial_u^{k-1} G_n(0)) \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{nk(n+1)} \times \mathbb{L}[[t]]^{nk}.$$

Then the saturated ideal $\langle \mathcal{S}_{dup} \rangle : \det(\mathsf{Jac}_{\mathcal{S}_{(dup)}})^\infty$ is zero-dimensional and radical over $\mathbb{L}(t)$. Moreover, $\mathcal{P}$ lies in the zero set of $\langle \mathcal{S}_{\mathsf{dup}} \rangle : \det(\mathsf{Jac}_{\mathcal{S}_{\mathsf{dup}}})^\infty$.

Therefore, in order to conclude the algebraicity of $G_i(0), \dots, \partial_u^{k-1} G_i(0)$ over $\mathbb{L}(t)$ for all $1 \le i \le n$, it is enough to justify that $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ is invertible at $\mathcal{P}$. Then, by Lemma 2, it will follow that it is possible to apply effective techniques from polynomial elimination theory and find annihilating polynomials for the power series of interest.

The idea for proving that $\det(\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}) \neq 0$, analogous to the proof in [4], is to show first that $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ rewrites as a block triangular matrix. We will then show that each such block is invertible by carefully analyzing its lowest valuation in t.

If $A \in \mathbb{L}[t][X_1, \dots, X_n, u]$, we shall define its "$i^{\text{th}}$ duplicated polynomial" as $A^{(i)} \coloneqq A(X_{ni+1}, \dots, X_{n(i+1)}, u_i)$. Then the Jacobian matrix $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ has the shape

$$\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})} = \begin{pmatrix} A_1 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & A_{nk} & B_{nk} \end{pmatrix} \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{nk(n+2) \times nk(n+2)},$$

where the matrices $A_i \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{(n+2)\times(n+1)}$ and $B_i \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{(n+2)\times nk}$ are given by:

$$A_i := \begin{pmatrix} \partial_{x_1} E_1^{(i)}(U_i) & \dots & \partial_{x_n} E_1^{(i)}(U_i) & \partial_{u_i} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^{(i)}(U_i) & \dots & \partial_{x_n} E_n^{(i)}(U_i) & \partial_{u_i} E_n^{(i)}(U_i) \\ \partial_{x_1} \mathsf{Det}^{(i)}(U_i) & \dots & \partial_{x_n} \mathsf{Det}^{(i)}(U_i) & \partial_{u_i} \mathsf{Det}^{(i)}(U_i) \\ \partial_{x_1} P^{(i)}(U_i) & \dots & \partial_{x_n} P^{(i)}(U_i) & \partial_{u_i} P^{(i)}(U_i) \end{pmatrix}, B_i := \begin{pmatrix} \partial_{z_0} E_1^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots \\ \partial_{z_0} E_n^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_n^{(i)}(U_i) \\ \partial_{z_0} \mathsf{Det}^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} \mathsf{Det}^{(i)}(U_i) \\ \partial_{z_0} P^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} P^{(i)}(U_i) \end{pmatrix}.$$

Using $\mathsf{Det}(U_i) = 0$ and (8), we see that the first $n \times (n+1)$ minor of each A_i has rank at most $n-1$. Hence, after performing operations on the first n rows, we can transform the n^{th} row of A_i into the zero vector. It follows that after the suitable transformation and a permutation of rows, $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ rewrites as a block triangular matrix. To give the precise form of the determinant of $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$, we first define

$$R := \det \begin{pmatrix} \partial_{x_1} E_1^{(i)}(U_i) & \dots & \partial_{x_{n-1}} E_1^{(i)}(U_i) & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^{(i)}(U_i) & \dots & \partial_{x_{n-1}} E_n^{(i)}(U_i) & y_n \end{pmatrix} \in \mathbb{K}[\{\partial_{x_\ell} E_j^{(i)}(U_i)\}_{1 \le \ell, j \le n}][y_1, \dots, y_n]. \tag{9}$$

Then it follows that $\det(\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}) = \pm\Big(\prod_{i=1}^{nk} \det(\mathsf{Jac}_i(U_i))\Big) \cdot \det(\Lambda)$, where

$$\mathsf{Jac}_i(u) := \begin{pmatrix} \partial_{x_1} E_1^{(i)}(u) & \dots & \partial_{x_n} E_1^{(i)}(u) & \partial_{u_i} E_1^{(i)}(u) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_{n-1}^{(i)}(u) & \dots & \partial_{x_n} E_{n-1}^{(i)}(u) & \partial_{u_i} E_{n-1}^{(i)}(u) \\ \partial_{x_1} \mathsf{Det}^{(i)}(u) & \dots & \partial_{x_n} \mathsf{Det}^{(i)}(u) & \partial_{u_i} \mathsf{Det}^{(i)}(u) \\ \partial_{x_1} P^{(i)}(u) & \dots & \partial_{x_n} P^{(i)}(u) & \partial_{u_i} P^{(i)}(u) \end{pmatrix} \in \mathbb{L}[u][[t]]^{(n+1)\times(n+1)}, \text{ and}$$

$$\Lambda := \left(R(\partial_{z_j} E_1^{(i)}(U_i), \dots, \partial_{z_j} E_n^{(i)}(U_i))\right)_{1 \le i, j+1 \le nk} \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{nk \times nk}. \tag{10}$$

The proof that this product is non-zero is the content of Lemma 3 and Lemma 4.

Lemma 3. *For each $i = 1, \dots, nk$, the determinant of $Jac_i(U_i)$ is non-zero.*

Proof. To prove that $\det(\mathsf{Jac}_i(U_i)) \neq 0$ we will show that $\mathsf{val}_t(\det(\mathsf{Jac}_i(U_i))) < \infty$, where val_t denotes the valuation in t. The main idea here is to expand $\det(\mathsf{Jac}_i(U_i))$ with respect to the last column and show that the least valuation comes from the product of $\partial_{u_i} \mathsf{Det}^{(i)}(U_i)$ by its associated minor which we denote by $\mathcal{M}$.

Since $\partial_{x_j} \mathsf{Det}^{(i)}(U_i) = O(t^\alpha)$, it is clear that for $1 \le j \le n$ the determinants of the minors associated to $\partial_{u_i} E_{j-1}^{(i)}(U_i)$ and $\partial_{u_i} P^{(i)}(U_i)$ are in $O(t^\alpha)$. It remains to show that the product of $\partial_{u_i} \mathsf{Det}^{(i)}(U_i)$ by $\det(\mathcal{M})$ is of valuation in t strictly lower than α. For $j = 1, \dots, n-1$ and $\ell = 1, \dots, n$ one computes that $\mathsf{val}_t(\mathcal{M})_{j,\ell} = \partial_{x_j} E_\ell^{(i)}(U_i) = \beta + \frac{m_j}{k}$ if $j \neq \ell$ and $\frac{m_j}{k}$ if $j = \ell$. Moreover, it follows from the definition of P and expansion along the last row of the matrix which defines P that the term with lowest t-valuation in $\partial_{x_n} P^{(i)}$ is given by the product of $\partial_{x_n, u_i} E_n^{(i)}$ by the determinant of the associated minor of $\partial_{u_i} E_n^{(i)}$. Computing this valuation while using that α, β are chosen sufficiently large, we find that $\mathsf{val}_t(\partial_{x_n} P^{(i)}) = (\sum_{i=1}^n \frac{m_i}{k}) - \frac{1}{k} = (M-1)/k$. It follows that the only monomial in the determinant of $\mathcal{M}$ that has no dependency on β comes from the product of diagonal elements of $\mathcal{M}$. Using the definition $\alpha = n^2k \cdot (\lfloor 2M/k \rfloor + 1) + nM$ and $\beta = \lfloor 2M/k \rfloor$, we conclude that $\mathsf{val}_t(\partial_{u_i}\mathsf{Det}^{(i)}(U_i) \cdot \det(M)) < \alpha$. □

Lemma 4. *The determinant of Λ is non-zero.*

Proof. Proving $\det(\Lambda) \neq 0$ is again done by analyzing the first terms of $\det(\Lambda)$. We prove can that mod t^α the determinant factors as a product of U_i, the Vandermonde determinant $\prod_{i<j}(U_i - U_j)$, and a non-zero polynomial $H(t)$. The actual computation is somewhat technical, since $\det(\Lambda)$ is defined as the determinant of the $nk \times nk$ matrix $R(\partial_{z_j}E_1^{(i)}(U_i), \dots, \partial_{z_j}E_n^{(i)}(U_i))_{i,j}$ whose entries are themselves determinants of the $n \times n$ matrices (9). We shall give an exposition of the proof, omitting technical details.

We denote $R_j(u) \coloneqq R(\partial_{z_j}E_1^{(i)}(u), \dots, \partial_{z_j}E_n^{(i)}(u))$ and compute $R_j(u_i) \bmod t^\alpha$ for variables $u_1, \dots, u_{nk}$. Note that the latter is a non-zero polynomial in $\mathbb{L}[u_1, \dots, u_{nk}, t]$ which is independent of the polynomials $Q_1, \dots, Q_n$. Let $\tilde{\Lambda} = (R_j(u_i))_{1 \le i, j+1 \le nk}$ be the matrix Λ with the U_i replaced by the variables u_i. With tedious but explicit computations it is possible to show that each element in the i^{th} row of $\tilde{\Lambda} \bmod t^\alpha$ is a polynomial in u_i of degree $\le M - 1$ and valuation $\ge M - nk$. Moreover, all entries of $\tilde{\Lambda} \bmod t^\alpha$ have degree in t bounded by $t^{n(\beta+1)}$. The choice for α and β ensures that $\det(\tilde{\Lambda} \bmod t^\alpha) = \det(\tilde{\Lambda}) \bmod t^\alpha$.

As we have $M \ge nk$, it is possible to factor out u_i^{M-nk} from the i^{th} row of $\tilde{\Lambda} \bmod t^\alpha$ when computing its determinant. This yields polynomials of degree at most $nk - 1$ in u_i on the i^{th} row. Moreover, it is obvious that if $u_i = u_j$ for some $i \neq j$, the determinant of $\tilde{\Lambda}$ vanishes. Hence, we can also factor out the Vandermonde determinant $\prod_{i<j}(u_i - u_j)$. As this latter product is of degree $nk - 1$ in u_i, we conclude that

$$\det(\tilde{\Lambda}) \equiv \prod_{i=1}^{nk} u_i^{M-nk} \cdot \prod_{i<j}(u_i - u_j) \cdot H(t) \mod t^\alpha, \tag{11}$$

for some non-zero polynomial $H \in \mathbb{L}[t]$ whose degree only depends on β. Recall that $\tilde{\Lambda}(U_1, \dots, U_{nk}) = \Lambda$, and all U_i are distinct with valuation in t of $1/k$ by Lemma 1. Using this, equation (11) and $\alpha > (M - nk)n + n + n^2k(\beta + 1)$, we conclude that $\det(\Lambda) \neq 0$. □

Having now proved that $\det(\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}) \neq 0$ at $\mathcal{P}$, we can apply Lemma 2 and obtain that the specialized series $G_i(0), \dots, \partial_u^{k-1}G_i(0)$ are all algebraic over $\mathbb{K}(t, \epsilon)$. The algebraicity of the complete formal power series $G_1, \dots, G_n$ over $\mathbb{K}(t, u, \epsilon)$ then follows again by [3, Lemma 2.10] from the invertibility the Jacobian matrix of $E_1, \dots, E_n$ considered with respect to the variables $x_1, \dots, x_n$ (with $t, u, z_0, \dots, z_{nk-1}$ viewed as parameters). The equalities $F_i(t^\alpha, u) = G_i(t, u, 0)$ finally imply that $F_1, \dots, F_n$ are also algebraic over $\mathbb{K}(t, u)$.

As already mentioned before, a strength of the presented method is that it is effective. Recall that Theorem 2 summarizes a bound on the algebraicity degree of all $F_i(t, u)$ and estimates the arithmetic complexity of the algorithm which computes $F_i(t, a)$.

Proof of Theorem 2. Using the definition of α and β in the proof of Theorem 1, the result is proven along the same lines as the results in [3, Section 3]. The algebraicity bound is a consequence of the Bézout theorem applied to the saturated ideal defined in Lemma 2, while the announced complexity is a consequence of [10, Theorem 2]. This proof yields the bound $(n^{n^2k^2} \cdot (n+1)^{n^2k^2} \cdot (k+1)^{n(3nk^2+1)} \cdot \delta^{n(3nk^2+1)})/(nk)!^{nk}$ on the algebraicity degree and also justifies that the arithmetic complexity of the presented algorithm is contained in $O(k^{10nk+2n+4} \cdot n^{27nk+6n+9} \cdot 2^{10nk+5n} \cdot \delta^{10nk+2n+3})$. $\square$

4 Summary and future work

We can summarize the strategy presented in Section 3 as follows:

1. Set up the deformed system (6) and the polynomials $\mathsf{Det}, P \in \mathbb{L}[X_1, \dots, X_n, t, u]$:

$$\mathsf{Det} := \det\begin{pmatrix} \partial_{x_1}E_1 & \dots & \partial_{x_n}E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1}E_n & \dots & \partial_{x_n}E_n \end{pmatrix} \quad \text{and} \quad P := \det\begin{pmatrix} \partial_{x_1}E_1 & \dots & \partial_{x_1}E_{n-1} & \partial_{x_1}E_n \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_{n-1}}E_1 & \dots & \partial_{x_{n-1}}E_{n-1} & \partial_{x_{n-1}}E_n \\ \partial_u E_1 & \dots & \partial_u E_{n-1} & \partial_u E_n \end{pmatrix}.$$

2. Set up the duplicated polynomial system $(\mathcal{S}_{\mathsf{dup}})$, consisting of the duplications of the polynomials E_i, Det, P. It has $nk(n+2)$ variables and equations.

3. Compute a non-trivial element of the saturated ideal $\langle \mathcal{S}_{\mathsf{dup}} \rangle : \det(\mathsf{Jac}_{\mathcal{S}_{\mathsf{dup}}})^\infty$.

As illustrated in Example 1, the deformation step is not always needed. In fact, it is clear that for a generic system the equation $\mathsf{Det}(u) = 0$ will have nk distinct non-zero solutions in $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$. Moreover, generically, a non-trivial element of both $\langle \mathcal{S}_{\mathsf{dup}} \rangle : \det(\mathsf{Jac}_{\mathcal{S}_{\mathsf{dup}}})^\infty$ and $\langle \mathcal{S}_{\mathsf{dup}} \rangle : (\prod_{i \neq j}(u_i - u_j))^\infty$ contains the seeked annihilating polynomial. In practice, however, the deformation is important, as the following example shows:

Example 2 (cont.). For s randomly chosen in $\mathbb{Q}$, one cannot apply Algorithm 1 because the ideal $\langle \mathcal{S}_0, \mathcal{S}_1, m \cdot (u_1 - u_2) - 1 \rangle$ is not 0-dimensional, despite the fact $\mathsf{Det}(u) = 0$ has 2 distinct solutions. However, as predicted by the theory, after the deformation (6) the system indeed becomes zero-dimensional and can be solved systematically, even though the actual computation becomes quite heavy.

Our strategy produces a polynomial system with $nk(n+2)$ variables. Since, already for small values n, k, such systems are often out of reach, we wish to briefly introduce an approach that has a better algorithmic complexity. The idea is to reduce (by eliminating $F_2, \dots, F_n$) the initial system to a single functional equation $R = 0$, and then to use Bousquet-Mélou and Jehanne's method [4]. This reduces to solving a polynomial system with just $3nk$ variables and equations. In order to make this approach work, there are two necessary conditions: the equation $\partial_{x_1} R = 0$ should contain enough (that is nk) roots in $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$ and the corresponding ideal should be zero-dimensional. Note that R is not a DDE anymore in general, so these conditions are not guaranteed. The following proposition ensures that our deformation takes care of the first part, and the example right after shows that the second condition can still fail in practice.

Proposition 1. *Let* $(\mathbf{E}_{\mathbf{F_1}}),\dots,(\mathbf{E}_{\mathbf{F}_n})$ *be as in Theorem 1 and suppose that* $E_1,\dots,E_n$ *are the polynomials obtained after deforming* $(\mathbf{E}_{\mathbf{F_1}}),\dots,(\mathbf{E}_{\mathbf{F}_n})$ *as in (7). Let* $U_1,\dots,U_{nk}\in\overline{\mathbb{K}(\epsilon)}[[t^{\frac{1}{\star}}]]$ *be the distinct non-zero series solutions in* u *of the equation* $\mathsf{Det}(u)=0$ *and let* $R\in(\langle\mathcal{S}_{dup}\rangle:\det(\mathsf{Jac}_{(\mathcal{S}_{dup})})^{\infty})\cap\mathbb{L}[x_1,z_0,\dots,z_{nk-1},t,u]$*. Then* $U_1,\dots,U_{nk}$ *are also solutions of* $\partial_{x_1}R(u)=0$*.*

Proof. Since $R\in\langle E_1,\dots,E_n\rangle$, there exist $V_1,\dots,V_n\in\mathbb{L}[x_1,\dots,x_n,z_0,\dots,z_{nk-1},t,u]$ such that $R(U_\ell)=\sum_{i=1}^n E_i(U_\ell)V_i(U_\ell)$ for any $\ell=1,\dots,nk$. Differentiating with respect to x_j for $j=1,\dots,n$ and using that $E_i(U_\ell)=0$ and that R does not depend on x_j for $j\geq 2$, we find

$$\begin{pmatrix}\partial_{x_1}R(U_\ell)\\ 0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}\partial_{x_1}E_1(U_\ell) & \dots & \partial_{x_1}E_n(U_\ell)\\ \vdots & \ddots & \vdots\\ \partial_{x_n}E_1(U_\ell) & \dots & \partial_{x_n}E_n(U_\ell)\end{pmatrix}\begin{pmatrix}V_1(U_\ell)\\ \vdots\\ V_n(U_\ell)\end{pmatrix}. \tag{12}$$

By definition of U_ℓ, the matrix $(\partial_{x_j}E_i(U_\ell))_{i,j}$ is singular and Lemma 1 implies that each of its $(n-1)\times(n-1)$ minors is invertible. It follows that we can express the first row of the matrix as a linear combination of the other rows, then (12) implies that $\partial_{x_1}R(U_\ell)=0$. □

Example 2 (cont.). For s randomly chosen in $\mathbb{Q}$, reducing to a single equation R (by taking the resultant with respect to x_2), we indeed find that $\partial_{x_1}R(u)=0$ has two distinct roots in $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$. However, the computation of a Gröbner basis reveals that the corresponding ideal has positive dimension.

Future work. The present work provides a fruitful toolbox for proving algebraicity constructively and elementarily. Practical experiments (which are also based on further algorithmic tools that are under development) make us believe that our method has good potential for practical unresolved combinatorial examples as well. Moreover, there are three most natural directions for further work. They will deal with complexity improvements for practical computations and theoretical generalizations:

1. Exploit the strategy *hybrid guess-and-prove*, which was used in [3, Section 2.2.2] to tackle first order scalar DDEs efficiently, and which turns out to be useful when dealing with huge polynomial systems.

2. Proposition 1 ensures that the deformation (6) guarantees nk distinct roots of $\partial_{x_1}R(u)=0$, however, as demonstrated above, the corresponding ideal might still have positive dimension. Investigate whether it is possible to overcome this issue.

3. Extend the results to a higher number of "nested" catalytic variables, where the algebraicity is still guaranteed by Popescu's theorem, but with no effective version.

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